Problem Seminar

March 9, 2009: $\mathbb{R}^N$ as a metric space. Sets in $\mathbb{R}^N$

Instructor: Constantin P. Niculescu

Geometry of the Euclidean space $\mathbb{R}^N$

1. Prove that in $\mathbb{R}^N$ an equality of the form $\|x+y\| = \|x\| + \|y\|$ is possible if and only if x and y lie on the same half-line that passes through the origin. In other words, the norm function is strictly convex.

2. (a) Prove that $\left\| \frac{x}{\|x\|^2} - \frac{y}{\|y\|^2} \right\| = \frac{\|x-y\|}{\|x\| \cdot \|y\|}$ for all $x, y \in \mathbb{R}^N$.

   (b) Infer the inequality of Ptolemy,
   $$\|x\| \, \|y-z\| \leq \|y\| \, \|z-x\| + \|z\| \, \|x-y\| \, .$$

   (c)* Discuss the equality case.

Questions marked with * are more involving.

3*. Prove that every point of a compact convex set in $\mathbb{R}^N$ is a convex combination of at most $N+1$ extreme points.

Elementary topology of $\mathbb{R}^N$

1. Prove that the closure of the ball

$$B_r(a) = \{x : \|x - a\| < r\}$$

is the ball $\bar{B}_r(a) = \{x : \|x - a\| \leq r\}$ and the interior of $\bar{B}_r(a)$ is $B_r(a)$.

2. Prove that $\mathbb{Q} \times \mathbb{Q}$ is neither closed nor open in $\mathbb{R}^2$ but it is dense.

3. Prove that every subset of $\mathbb{R}^N$ is separable. Infer that every open covering of a subset of $\mathbb{R}^N$ contains a countable subcovering.

4. Prove that the following properties are equivalent for K a subset of $\mathbb{R}^N$:

   (a) K is compact;

   (b) K is bounded and closed;

   (c) Every sequence of elements of K contains a convergent sequence (with limit in K).

5. Prove that the interior and the closure of a convex set is convex too.

6. Prove that the intersection of every countable family of dense open subsets of $\mathbb{R}^N$ is dense.

7. List all subsets of $\mathbb{R}^N$ which are both open and closed.

8. Consider the space $\mathsf{M}_n(\mathbb{R})$ of all $n \times n$ dimensional real matrices, endowed with the topology induced by $\mathbb{R}^{n \times n}$. Prove that the subset of all invertible matrices is open.

References

[1] J. Jost, *Postmodern Analysis*, 2nd ed., Springer Verlag, 2003.

[2] Constantin P. Niculescu, *An Introduction to Mathematical Analysis*, Universitaria Press, Craiova, 2005.

[3] C. P. Niculescu and L.-E. Persson, *Convex Functions and their Applications. A Contemporary Approach.* CMS Books in Mathematics **23**, Springer Verlag, 2006.

[4] M. H. Protter and C. B. Morrey, *A First Course in Real Analysis*, 2nd ed., Springer Verlag, 1991.

[5] W. Rudin: *Principles of Mathematical Analysis*, 3rd Edition, McGraw-Hill Book Co., New York, 1976.