Problem Seminar

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Instructor: Constantin P. Niculescu

Finding the extrema

1. Find the extrema of the following function

$$u(x,y) = x^2 + y^2, \text{ for } 4x^2 + y^2 \leq 1.$$

2. Determine the smallest value of each of the following functions:

   (a) $u\left(x,y\right) = x^2 + xy + y^2 - 3x - 3y$ for $x, y \in \mathbb{R}$;

   (b) $u(x,y,z) = \frac{x^3+y^3+z^3}{3xyz}$, for $x, y, z > 0$.

Questions marked with * are more involving.

3. Determine the maximum of the following function:

   $u(x, y) = \sin x + \sin y + \sin(x + y)$, for $x, y \in [0, \pi/2]$;

4. Find the extrema of the function

$$u(x, y, z) = \sin \frac{x}{2} \sin \frac{y}{2} \sin \frac{z}{2},$$

   for $x + y + z = \pi$, $x > 0$, $y > 0$, $z > 0$.

5. (Hadamard's inequality). Prove that

$$u = \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \leq 1$$

   if $a_1^2 + b_1^2 + c_1^2 = 1$, $a_2^2 + b_2^2 + c_2^2 = 1$, $a_3^2 + b_3^2 + c_3^2 = 1$.

6. (a) Of all triangles of a given perimeter, find the triangle that has the greatest area.

(b)* (Dido's problem) The figure bounded by a line which has the maximum area for a given perimeter is a semicircle. Find a solution in the class of smooth functions.

Extrema of convex functions

7*. Find the maximum of the function

$$u(x,y,z) = 3(x^5 + y^7 \sin \frac{\pi x}{2} + z) - 2(xy + yz + zx)$$

for $x, y, z \in [0,1]$.

8. (Minkowski's inequality). Prove that

$$\left(\prod_{k=1}^{n} (x_k + y_k) \right)^{1/n} \geq \left(\prod_{k=1}^{n} x_k \right)^{1/n} + \left(\prod_{k=1}^{n} y_k \right)^{1/n}$$

for every $x_1, ..., x_n, y_1, ..., y_n \geq 0$. When does equality occur?

Weierstrass theorem in noncompact domains

9. Let $u : \mathbb{R}^N \to \mathbb{R}$ be a continuous function such that $\lim_{\|x\| \to \infty} f(x) = \infty$. Prove that f admits a global minimum.

References

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[4] M. H. Protter and C. B. Morrey, *A First Course in Real Analysis*, 2nd ed., Springer Verlag, 1991.

[5] W. Rudin: *Principles of Mathematical Analysis*, 3rd Edition, McGraw-Hill Book Co., New York, 1976.