The Recurrent Sequences as Dynamical Systems

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A *discrete dynamical system* on a metric space M is a sequence (f^n) of iterations of a function $f : M \to M$. Precisely, f^0 is the identity function and for $n \geq 1$,

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

The main problem: given an initial state $a \in M$ what happens with orbit of a,

$$x_0 = a, \quad x_n = f^{(n)}(x_0) = f(x_{n-1}) \quad \text{for} \quad n \geq 1?$$

Dynamical Systems Theory deals with the global study of orbits. This theory was initiated by H. Poincaré in connection with his research on Celestial Mechanics.

Invariant sets. Attractors

In what follows $f : M \to M$ is a function acting on a metric space M. If $a \in M$ and $f^n(a) = a$ for some $n \in \mathbb{N}^*$, then a is called a *periodic point* of f (of period n). The smallest such n is the *main period of* a. The orbit of a periodic point is a finite set. It can consist of only one point if a is a *fixed point* of f, that is if $f(a) = a$.

For the identity function of $\mathbb{R}$ all the points are fixed points. For the function $f(x) = -x$ there is only one fixed point, 0, all the other being periodic of period 2.

The fixed points and the periodic orbits are example of invariant sets. A subset A of M is called *invariant* under f (or for f) if $f(A) = A$ and is called *positively invariant* if $f(A) \subset A$. In both cases, any orbit starting in A stays in A.

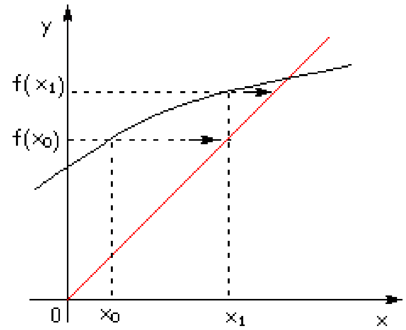

Figure 1: The step diagram.

When M is an interval, the dynamic can be revealed by the *step diagram*.

The intersection of the graph with the first bisector gives the fixed points. The sequence $(f^n(a))$ of the iterations of $x_0 = a$ can be seen in the following way: draw a vertical line through a, It will cross the graph at $(a, f(a))$. The horizontal line through this point will intersect the first bisector at $(f(a), f(a))$. The vertical line through this point will cross the graph at $(f(a), f^2(a))$ and so on. On the picture, the arrows show the order while the steps show the monotony of the function.

Examples. (a) Let $f : [-1, \infty) \to [-1, \infty)$, $f(x) = \sqrt{1+x}$. This function has an unique fixed point, $p =$

$\frac{1+\sqrt{5}}{2}$. The analysis of the graph shows that if $x_0 < p$ then $f^n(x_0)$ is a sequence increasing to p while if $x_0 > p$ then $f^n(x_0)$ is a sequence decreasing to p. See Figure 2.

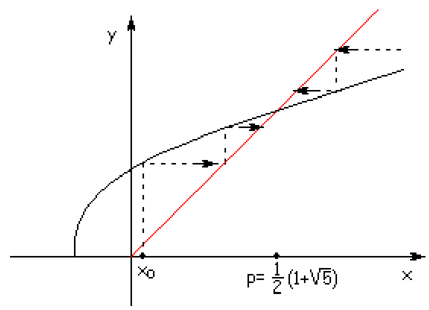

Figure 2: Dynamics of $f(x) = \sqrt{1+x}$.

(b) Let $f_\lambda : \mathbb{R} \to \mathbb{R}$, $f_\lambda(x) = \lambda x$. The origin is the only fixed point for f_λ, unless $\lambda = 1$. If λ changes a little bit then the dynamic of the points close to 0 can change considerably. See Figure 3.

If $\lambda = 0$ then $f^n(x_0) = 0$ for every x_0 and every $n \geq 1$.

If $\lambda \in (0,1)$ then all trajectories are converging to 0 (increasing, if $x_0 < 0$ and decreasing if $x_0 > 0$).

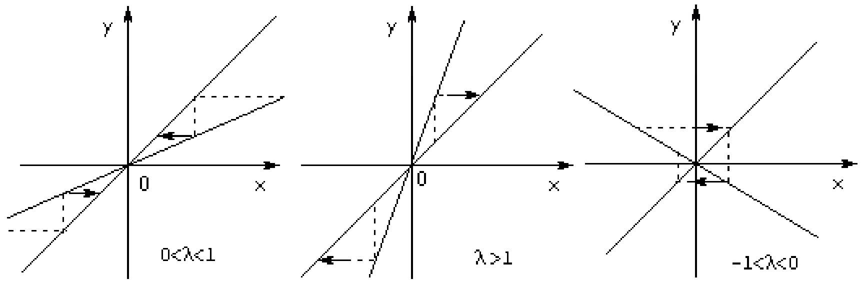

Figure 3: Dynamics of $f_\lambda(x) = \lambda x$.

If $\lambda = 1$ all the trajectories are constant.

If $\lambda > 1$ then for $x_0 > 0$ the trajectory converges (increasingly) toward ∞ and for $x_0 < 0$ the trajectory converges (increasingly) toward $-\infty$.

If $\lambda < -1$ then all trajectories are going away from the origin.

If $\lambda = -1$ then every point is periodic of period 2.

If $\lambda \in (-1, 0)$ then all trajectories are converging to 0.

These above examples show that there are many types of fixed points.

A fixed point p is called *attractor* for the dynamical system $f : M \to M$ if there is an open neighborhood U of p in M such that $f^n(x_0) \to p$ for all $x_0 \in U$. The set U is called the *basin of attraction* of f. If we can take $U = M$ then p is said to be a *global attractor*.

A fixed point p is said to be a *repellor* if there is a neighborhood U of p such that for every $x_0 \in U$, there is n with $f^n(x_0) \notin U$. Notice that in a case like this it may be possible that for some $m > n$, $f^m(x_0) \in U$.

There are *indifferent* fixed points, which are neither attractive nor repelling. For example 0 for the identity function of $\mathbb{R}$.

An important sources of global attractors is the Contraction Principle. Recall that a function $f : M \to M$ (here M is a metric space with metric d) is said to be a *contraction* if there is $C \in [0, 1)$ such that

$$d(f(x), f(y)) \leq Cd(x, y)$$

for every $x, y \in M$.

Theorem A (The Contraction Principle). *Let M be a complete metric space and $f : M \to M$ be a contraction. Then f has an unique fixed point which is a global attractor.*

Proof. Let $x_0 \in M$ and $x_n = f^n(x_0)$. For $n \geq 1$,

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq C d(x_n, x_{n-1})$$

and, using induction, we get that

$$d(x_{n+1}, x_n) \leq C^n d(x_1, x_0).$$

By the triangle inequality,

$$\begin{aligned}
&d(x_{n+k}, x_n) \\
&\leq d(x_{n+k}, x_{n+k-1}) + d(x_{n+k-1}, x_{n+k-2}) + \cdots + d(x_{n+1}, x_n) \\
&\leq C^{n+k-1} d(x_1, x_0) + C^{n+k-2} d(x_1, x_0) + \cdots + C^n d(x_1, x_0) \\
&= C^n (C^{k-1} + C^{k-2} + \cdots + 1)\, d(x_1, x_0) \\
&\leq \frac{C^n}{1 - C}\, d(x_1, x_0).
\end{aligned}$$

Since the last quantity converges to 0 as n approaches ∞ we conclude that $(x_n)_n$ is a Cauchy sequence and thus a convergent one because the space M is complete. Let p be the limit of the sequence. Since $x_n \to p$ and f is a continuous function, $f(x_n) \to f(p)$ which means $x_{n+1} \to f(p)$, from where we conclude that $f(p) = p$. Thus f has a fixed point. Assume now that there is another fixed point q. Then

$$d(p,q) = d(f(p), f(q)) \leq Cd(p,q).$$

If $d(p,q) \neq 0$ then we can cancel it to get $C \geq 1$, which is a contradiction. Hence $d(p,q) = 0$ and so $p = q$. Thus every orbit converges to a fixed point and there is only one fixed point, which means that every orbit converges to the same point. ■

This theorem is also known as the Banach-Cacciopoli Fixed Point Theorem.

The Contraction Principle gives us a way of finding the fixed point, which is called the *method of successive approximations*. Start with any x_0 and keep applying f. The sequence $f^n(x_0)$ converges to the fixed point.

The Contraction Principle also gives us an evaluation of the error. Looking back in the proof, we see

$$d(x_{n+k}, x_n) \le \frac{C^n}{1-C} d(x_1, x_0).$$

When $k \to \infty$ we obtain

$$d(p, x_n) \le \frac{C^n}{1-C} d(x_1, x_0).$$

An interesting history of the method of successive approximations can be found in D. F. Bailey [2]. The method originates in antiquity, connected to the Babylonian algorithm of computing the square root of a positive number.

Exercises

1. Use the step diagram to discuss the dynamic of the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = -x + 3x^2 - x^3.$$

2. The method of successive approximations is *more general* than the Contraction Principle. In other words, the method may work even for functions that are not contractions. Here is an example. Prove that the sequence

$$x_0 = a \quad x_{n+1} = \sin x_n \quad \text{for } n \geq 0$$

converges to 0, the unique fixed point of the function $\sin x$.

3. (The Babylonian Algorithm for computing roots).

(a) Prove that the sequence $(a_n)_n$ defined by

$$a_0 = 3 \quad \text{and} \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{8}{a_n}\right)$$

is convergent to $\sqrt{8}$.

(b) Notice that $a_{n+1}^2 - 8 \leq \left(a_n^2 - 8\right)/32$, and thus

$$0 < a_n^2 - 8 < 32^{1-2^n}$$

for all $n \in \mathbb{N}$. Then infer that $0 < a_n - \sqrt{8} < 10 \cdot 10^{-3 \cdot 2^{n-2}}$, for all $n \in \mathbb{N}$.

(c) Extend the result of (a) by showing that for every $p \in \mathbb{N}$, $p \geq 2$, and every $a > 0$ the sequence $(a_n)_n$ given by

$$a_{n+1} = \frac{1}{p}\left((p-1)a_n + \frac{a}{a_n^{p-1}}\right)$$

is convergent to $\sqrt[p]{a}$, whenever is $a_0 > 0$.

4. (Gauss' Arithmetic-Geometric Mean). Let a and b two positive numbers, with $a \leq b$. Prove that the sequences defined by

$$x_0 = a, \quad y_0 = b,$$
$$x_{n+1} = \sqrt{x_n y_n}, \quad y_{n+1} = \frac{x_n + y_n}{2} \quad \text{for } n \geq 0$$

are convergent and they have a common limit $M(a, b)$. Follow the next steps:

(a) $y_n \geq x_n$ for all n.

(b) The sequence $(x_n)_n$ is increasing, while the sequence $(y_n)_n$ is decreasing. Conclude from here that both are convergent.

(c) Take the limit of the recursive relation $y_{n+1} = (x_n + y_n)/2$ to obtain that the two sequences $(x_n)_n$ and $(y_n)_n$ have the same limit.

5. (B. P. Hillam [2]). Let $f : [a,b] \to [a,b]$ be a Lipschitz function (of constant C) and $F : [a,b] \to [a,b]$, $F(x) = (1-\lambda)x + \lambda f(x)$, where $\lambda = 1/(1+C)$. Prove that F is increasing and that for any $x_0 \in [a,b]$ the sequence $x_n = F^n(x_0)$ converges to a fixed point of f.

6. (R. L. Franks and R. P. Marzec [2]). Let $f : [a,b] \to [a,b]$ be a continuous function. Prove that for any $x_0 \in [a,b]$ the sequence

$$x_{n+1} = \frac{nx_n + f(x_n)}{n+1}$$

converges to a fixed point of f.

Sharkovsky's Theorem

Theorem (Weak form of Sharkovsky's Theorem). *If a continuous function $f : \mathbb{R} \to \mathbb{R}$ has a periodic point of main period 3, then f has periodic points of any main period.*

Proof. This can easily done following the next steps:

Step 1: If $f : [a, b] \to \mathbb{R}$ is a continuous function and B is a subinterval of $f([a, b])$, then there is a compact subset C of $[a, b]$ such that $f(C) = B$.

Step 2: If $f : [a, b] \to \mathbb{R}$ is a continuous function and A is a nonempty compact interval such that $A \subset f(A)$, then f has a fixed point.

Step 3: Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that there is $c \in [a, b]$ with $f(a) = c, f(c) = b, f(b) = a$. Let $n \geq 2$ and $I_0 = I_1 = \cdots = I_{n-2} = I_n = [a, b]$

and $I_{n-1} = [a, c]$. There is a decreasing family $(A_k)_{k=1}^n$ of compact subintervals of $[a, b]$ such that $f^k(A_k) = I_k$ for all $k = 0, 1, 2, \ldots, n$. By Step 2, f^n has a fixed point in A_n. Thus f has a periodic point of (main) period n.

Step 4: From the previous steps it is now clear that the existence of a point of main period 3 implies the existence of periodic points of any main period. ■

In fact Sharkovsky proved a more powerful result, still specialized to intervals. Consider the Sharkovsky order on $\mathbb{N}^*$:

$$3 \triangleleft 5 \triangleleft 7 \cdots \triangleleft 2{\cdot}3 \triangleleft 2{\cdot}5 \triangleleft 2{\cdot}7 \cdots \triangleleft 2^2{\cdot}3 \triangleleft 2^2{\cdot}5 \triangleleft 2^2{\cdot}7 \cdots \triangleleft 2^3 \triangleleft 2^2 \triangleleft 2 \triangleleft 1.$$

First there are the odd numbers (without 1), then the same numbers multiplied by 2, then the same number multiplied by 4 and so on. At the end there are the power of 2 in decreasing order (including $2^0 = 1$).

Sharkovsky's Theorem). *If $m \triangleleft n$ and f has a periodic point of main period m, then f has a periodic point of main period n.*

For details, see R. Devaney [5], pp. 60-68.

Exercises

1. Consider the piecewise linear map $f : [1, 5] \longrightarrow [1, 5]$ for which $f(1) = 3$, $f(2) = 5$, $f(3) = 4$, $f(4) = 2$ and $f(5) = 1$. Prove that f has points of period 5 but no points of period 3. This example can be easily adapted to an example of a piecewise linear map $f : [1, 2n + 1] \longrightarrow [1, 2n + 1]$, that has points of period $2n + 1$ but no points of period $2n - 1$.

2. Give an example of a piecewise linear map $f : [1, 4] \longrightarrow [1, 4]$ that has points of period 2^2 but no points of period 2^3.

3. (J.P. Delahaye). Let $I = [0, 1]$ and $I_k = [1 - 1/3^k, 1 - 2/3^{k+1}]$, for all $k \geq 0$. For each k let $f_k : I_k \longrightarrow I_k$ be a continuous map. Define a continuous map $f : I \longrightarrow I$ by letting $f(1) = 1$, $f(x) = f_k(x)$ if $x \in I_k$ and extend it by linearity elsewhere. Prove that f has points of periods 2^n for all $n \geq 0$, but no points of other periods.

Hyperbolicity

In this section we will present an easy-to-check condition, due to Perron, which implies that a fixed point is an attractor or a repellor.

Definition. A periodic orbit $\mathcal{O}(p)$ of a C^1 map $f : I \to I$ is *hyperbolic* if

$$\left|(f^m)'(p)\right| \neq 1$$

where m is the main period of p.

An easy computation shows that this condition does not really depend on the point generating the periodic orbit. In fact, by the Chain Rule,

$$\begin{aligned} &(f^m)'(f^k(p)) \\ &= f'(f^{m-1}(f^k(p))) \cdot f'(f^{m-2}(f^k(p))) \cdots f'(f^k(p)) \\ &\qquad\qquad = f'(f^{m-1}(p)) \cdot f'(f^{m-2}(p)) \cdots f'(p) \end{aligned}$$

for all $k \in \{0, ..., m-1\}$, which yields

$$(f^m)'(x) = (f^m)'(p) \quad \text{for all } x \in \mathcal{O}(p),$$

How an orbit approaches a closed set is made precise by using the distance of a point x to a set A, which is defined by the formula

$$d(x, A) = \inf \{|x - a| : a \in A\}.$$

The next result shows that the hyperbolic orbits (of interval maps) are either attractors or repellors.

Theorem B. *Suppose that f is a C^1-map acting on an interval I and let p be a periodic point of main period m.*

(a) *If $|(f^m)'(p)| < 1$, then there is an open neighborhood U of $\mathcal{O}(p)$ such that $f(\overline{U}) \subset U$ and for each x of U we have*

$$\lim_{n \to \infty} d\left(f^n(x), \mathcal{O}(p)\right) = 0.$$

(b) *If $|(f^m)'(p)| > 1$, then there is an open neighborhood V of $\mathcal{O}(p)$ such that for each x in $V \setminus \mathcal{O}(p)$, one can find a positive integer n for which $f^n(x) \notin V$.*

Proof. For the sake of simplicity we shall consider here only the case of fixed points.

(a) Since f has continuous derivative and $|f'(p)| < C < 1$, there is $\varepsilon > 0$ such that

$$\left|f'(x)\right| < C \quad \text{on} \quad U = (p - \varepsilon,\, p + \varepsilon) \cap M.$$

According to the Mean Value Theorem, for every x in $\overline{U}$ we have

$$|f(x) - p| = |f(x) - f(p)| \leq C\,|x - p| < \varepsilon.$$

Therefore $f(x)$ belongs to U and, by iterating the above argument, we get

$$|f^n(x) - p| \leq C^n\,|x - p| \quad \text{for all } n \in \mathbb{N},$$

hence $f^n(x) \to p$.

The assertion (b) can be argued in a similar manner. ∎

For example, the function

$$f:\mathbb{R}\to\mathbb{R},\qquad f(x)=-\frac{x^3+x}{2}$$

has 0 as a hyperbolic attractor, with the basin of attraction $U=\mathbb{R}\setminus\{-1,1\}$. The periodic orbit $\{-1,1\}$ is a hyperbolic repellor.

We might think that a simple map necessarily has a simple dynamics. This is not quite so, and a first counterexample is provided by the problem to determine the domain of attraction. To be more specific, let us consider the odd map $f(x)=\left(3x-x^3\right)/2$, $x\in\mathbb{R}$. It has three hyperbolic fixed points: -1 and 1 are attractors, while the origin is a repellor. Trying to determine the domain of attraction of 1, the graphical analysis shows that this set contains the interval $(0,\sqrt{3})$; notice that $f(\sqrt{3})=0$. See Figure 4.The domain of attraction of 1 is actually the union of intervals

$$\bigcup_{n=0}^{\infty} f^{-n}\left((0,\sqrt{3})\right)=(0,\sqrt{3})\cup[-2,-\sqrt{3})\cup\ldots\ .$$

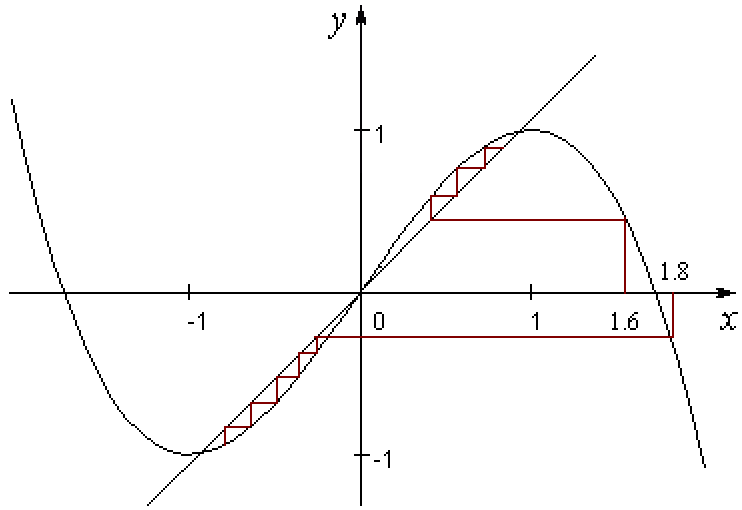

Figure 4: Nearby points iterate to different attractors.

The length of the components gets smaller with n. They are all contained in $(-\sqrt{5}, \sqrt{5})$, and this interval decomposes into the domain of attraction of 1, the domain of attraction of -1 and the repellor 0. Notice that $\left\{-\sqrt{5}, \sqrt{5}\right\}$ is a hyperbolic periodic repellor.

Exercises

1. Prove the following strengthening of part (b) of Theorem B: If $|f'(p)| > 1$ then there are neighborhoods V_1 and V_2 of p, and $n_0 \in \mathbb{N}^*$ such that $V_1 \subset V_2$ and $f^n(x) \notin V_2$ for all $x \in V_2 \setminus V_1$ and $n \geq n_0$.

2. Determine the nature of the fixed points of the following functions defined on $\mathbb{R}$:

$$f_1(x) = \sin x$$

$$f_2(x) = x^3 - x$$

$$f_3(x) = \arctan x$$

$$f_4(x) = e^x$$

Sensitive Dependence of Initial Conditions

By iterating a function $f : M \to M$ we may have big surprises.

One example: *doubling the angle map,*

$$f : S^1 \to S^1, \qquad f(z) = z^2.$$

Definition (J. Guggenheimer). A discrete dynamical system $f : M \to M$ is called to have *sensitive dependence of initial conditions* if there is $\delta > 0$ such that for every $a \in M$ and every V neighborhood of a, there is $y \in V$ and $n \in \mathbb{N}$ such that

$$d(f^n(x), f^n(y)) > \delta.$$

Definition. A dynamical system is called *chaotic* (in the sense of Devaney) if it satisfies the following three conditions:

(T) topological transitivity (i.e., the existence of points with dense orbits).

(P) the existence of a dense set of periodical points.

(S) sensitive dependence of initial conditions.

This definition had a great impact in the popularization of the theory of chaotic dynamical systems. Nevertheless, the three requirements are not independent.

Theorem C (J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey [3]).

$$(T)\,\&\,(P) \Rightarrow (S).$$

An interesting result in topological dynamics on $\mathbb{R}$ is as follows:

Theorem D (L. S. Block and W. A. Coppel [4]). *If I is a nondegenerate interval, then any continuous topological transitive function $f : I \longrightarrow I$ is chaotic.*

Chaos through *semiconjugation*:

Theorem E. *Let X and Y be metric spaces and the commutative diagram of continuous functions*:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h \downarrow & & \downarrow h \\ Y & \xrightarrow[g]{} & Y \end{array}$$

If f is chaotic and h is onto then g is chaotic.

Proof. It is easy to see that the properties (T) & (P) of f are transported to g. Then apply Theorem C. ■

An intriguing case of chaotic behavior is the Chebyshev polynomial of order 2,

$$T_2(x) = 2x^2 - 1 \quad \text{for} \quad x \in [-1, 1].$$

If we substitute $x = \cos\theta$ we get $T(x) = \cos(2\theta)$.

The chaotic behavior of T_2 follows from Theorem E applied for $X = S^1$, $Y = [-1, 1]$, $f =$ doubling the angle map, $h(e^{i\theta}) = \cos\theta$ (the projection onto the first component) and $g = T_2$.

From the chaotic behavior of T_2, we can infer (using as vertical map $h(x) = \frac{1}{2}(1 - x)$) the chaotic behavior of the logistic function

$$F_4 : [0, 1] \to [0, 1], \quad F_4(x) = 4x(1 - x).$$

The Logistic Family

The dynamics of the logistic family

$$F_\lambda(x) = \lambda x(1 - x), \quad x \in \mathbb{R}$$

for $\lambda > 1$ is presented in great details by Devaney [5]. As pointed out by R. May [13], this family appears very naturally in connection with an idealized model for the evolution of a biological population over time; x is the population density in a certain generation (confined to a certain environment), and $F_\lambda(x)$ is the population density of the next generation. $F_\lambda(x) = 0$ means extinction and $F_\lambda(x) = 1$ means saturation.

The critical zone for the behavior of the iterates of F_λ is $[0, 1]$.

The map F_λ has two fixed points: the origin 0 and $p_\lambda = 1 - 1/\lambda$. The origin is a hyperbolic repellor, while p_λ is an attractor for $1 < \lambda \leq 3$, with $(0, 1)$ as a domain of attraction; p_λ is hyperbolic only for $\lambda \in (1, 3)$.

When λ passes $\lambda_1 = 3$, p_λ becomes a hyperbolic repellor and a hyperbolic attracting orbit of period 2 is created. The next critical value is $\lambda_2 = 1 + \sqrt{6}$. Past λ_2, the latter periodic orbit becomes a hyperbolic repellor and a hyperbolic attracting orbit of period 2^2 is created. Increasing λ, a cascade of periodic orbits (with doubling periods, $2, 2^2, 2^3, ...$) occurs. The corresponding sequence of bifurcation values, $(\lambda_n)_n$, converges to a value $\lambda_\infty = 3.569946....$ The ratios of the distances between successive bifurcations converge too,

$$\lim_{n \to \infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} = \delta = 4.66920...$$

If $\lambda > \lambda_\infty$, then the dynamics becomes more irregular. For some values of λ there exist periodic orbits, which in turn give rise to new sequences of period doubling bifurcations, in a succession motivated by the Sharkovsky Theorem.

For $\lambda^\star = 1 + 2\sqrt{2} = 3.\,828\,4...$, an attracting periodic orbit of period 3 appears; see Exercise 4.

It can be easily visualized with a computer by iterating (for a sufficiently large number of steps) any point of $(0,1)$.

$$\begin{array}{rr} \alpha = 0.159\,9... & F_{\lambda^\star}(\alpha) = 0.514\,3... \\ F^2_{\lambda^\star}(\alpha) = 0.956\,3... & F^3_{\lambda^\star}(\alpha) = 0.159\,9... \end{array}$$

According to Sharkovsky's Theorem, F_λ admits periodic orbits of any period. However, being repelling, they do not appear on the computer screen!

For larger values of λ (but not for all of them) there exist orbits which are dense in the whole interval [0,1]. In fact, it was noticed by Graczyk and Swiatek [8] and Lyubich [9], [10] that:

- The set $\mathfrak{R}$ of values of $\lambda \in [1,4]$ for which F_λ has a periodic attractor is open and dense;

- Except for a Lebesgue negligible subset of $[1,4] \setminus \mathfrak{R}$, the map F_λ has a dense orbit.

The dynamics of F_λ is truly puzzling for $\lambda = 4$. In this case we encounter both the density of the set of periodic points and the presence of points with dense orbits. As an effect, the dynamics shows the phenomenon of *sensitive dependence on initial conditions*: trajectories starting very close together will rapidly separate, and thereafter have totally different futures.

For $\lambda > 4$, the maximum of F_λ is greater than 1 and some points escape $[0, 1]$ under iteration. However, there still exists an invariant set Δ_λ in [0,1] on which the dynamics of F_λ has the same features: density of periodic points, existence of dense orbits and sensitivity to the initial conditions. The only difference to the case $\lambda = 4$ is that Δ_λ is thin (homeomorphic to a Cantor set).

Exercises

1. Prove that all Chebyshev polynomials,

$$T_n(x) = \cos(n \arccos x) \quad (n \geq 2)$$

for $x \in [-1, 1]$, are chaotic.

2. Prove that $f(x) = \pi \sin x$ for $x \in [0, \pi]$ is chaotic.

3. Prove that the smallest value of $\lambda \in (0, 4)$ for which the logistic map F_λ has a period-3 orbit is $\lambda = 1 + 2\sqrt{2} = 3.828\,42\ldots$.

Evaluate the derivative of the third iterate F_λ^3 along this orbit $\mathcal{O}$ and conclude that $\mathcal{O}$ is an attractor.

[*Hint* (cf. [7]): The period orbit $x(n)$ can be written as $x(n) = \alpha + \beta\omega^n + \overline{\beta}\overline{\omega}^n$, where ω is a complex cube root of unity; α and β are complex parameters to be determined from the equation $x(n + 1) =$

$\lambda x(n)(1 - x(n))$. As $\omega^2 = \overline{\omega}$ and $\overline{\omega}^2 = \omega$, we are led to a system of equations which gives us

$$\alpha = \frac{3\lambda + 1 \pm \sqrt{\lambda^2 - 2\lambda - 7}}{6\lambda}.$$

So the smallest possible value of λ for which period-3 orbits are possible is the positive root of $\lambda^2 - 2\lambda - 7$, that is, $\lambda = 1 + 2\sqrt{2}$.]

4. The tent map $T_\lambda : [0, 1] \to \mathbb{R}$ ($0 < \lambda < \infty$) is defined by the formula

$$T_\lambda(x) = \lambda \left(\frac{1}{2} - \left| \frac{1}{2} - x \right| \right).$$

Suppose that if $\lambda > 2$. Prove that:

(a) the tent map has an invariant Cantor set Λ on which it is chaotic;

(b) all the orbits of $T_{\lambda|\Lambda\cap\mathbb{Q}}$ are eventually periodic, yet $T_{\lambda|\Lambda\cap\mathbb{Q}}$ is chaotic.

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